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Duality twists, orbifolds, and fluxes

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Abstract: We investigate compactifications with duality twists and their relation to orbifolds and compactifications with fluxes. Inequivalent compactifications are classified by conjugacy classes of the U-duality group and result in gauged supergravities in lower dimensions with nontrivial Scherk-Schwarz potentials on the moduli space. For certain twists, this mechanism is equivalent to introducing internal fluxes but is more general and can be used to stabilize some of the moduli. We show that the potential has stable minima with zero energy precisely at the fixed points of the twist group. In string theory, when the twist belongs to the T-duality group, the theory at the minimum has an exact CFT description as an orbifold. We also discuss more general twists by nonperturbative U-duality transformations.

Keywords: String Duality, Superstring Vacua.

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1. Introduction

In this paper we investigate compactifications that include duality twists and internal fluxes and their relation to orbifolds.

Compactification with duality twisting is a generalization of the Scherk-Schwarz mechanism in classical supergravity [1]–[13]. In a typical supergravity theory, there is a non-compact global symmetry G. In a twisted compactification, one introduces a twist in the toroidal directions by the global symmetry G. The twisting generates a nontrivial Scherk-Schwarz potential on the moduli space and for certain twists is equivalent to introducing internal fluxes of various gauge fields on the torus.

We consider the extension of duality twisting to the full quantum string theory and discuss the general properties of the resulting Scherk-Schwarz potential. The global symmetry G of the low energy effective action is not a symmetry of the quantum theory but is broken to a discrete U-duality group $G(\mathbb{Z})$ [14] that acts on the integral lattice of p-brane charges. Therefore, the twists that can be lifted to string theory must belong to the duality group $G(\mathbb{Z})$ [8]. This restriction leads to a quantization condition on the mass parameters appearing in the Scherk-Schwarz potential [8].

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As we review in section 2.1, the physically inequivalent twists are classified by conjugacy classes of $G(\mathbb{Z})$. We analyze the Scherk-Schwarz potential and show that the effective low energy physics of the compactified theory is completely determined by the conjugacy class, resolving an apparent paradox. Given a potential on the moduli space, the next question is whether the potential has any minima and what the structure of the theory is at these minima.

We will see that the task of finding the minima is simplified considerably by some elegant group theoretic considerations. We illustrate this point in section 3 by means of an explicit example in which the duality twists belong to $\mathrm{SL}(2,\mathbb{Z})$ and outline the generalization to other groups in section 4.4. We show that the minima of the Scherk-Schwarz potential are in one-to-one correspondence with the fixed points in the moduli space under the action of the twist group. One implication of this result is that for a compactification twisted by an element of the T-duality group, the theory at a minimum of the potential has an exact conformal field theory description as an orbifold of a toroidal compactification. The orbifold theory as usual contains additional twisted sector states that are not visible in the supergravity analysis. When the twist is *not* a perturbative symmetry, there is no CFT construction for such theories, but the supergravity analysis and the group theoretic considerations concerning the minima of the potential can still be applicable.

One motivation for this work is its bearing on the stabilization of moduli in string theory. The vacuum manifold of string compactifications is characterized by several moduli that govern the shape and size of the compactification space as well as the value of the coupling constant in string theory and correspond to unwanted massless fields in space-time. There are stringent observational constraints on the presence of such massless scalars and even in a cosmological context the presence of moduli is problematic [15]. It is thus interesting to seek string compactifications with few or no moduli already at the tree level.

A number of apparently unrelated methods have been utilized in the literature for constructing models with a small number of moduli. Compactifying with duality twists or internal fluxes is one way to stabilize the moduli. In this framework, the twists or the fluxes generate a nontrivial potential on the moduli space. As a result, the expectation values of the moduli fields are fixed at the minima of the potential and many moduli acquire mass [1]–[29]. This mechanism has been used, for example, to construct models where all complex structure moduli of Calabi-Yau and torus compactifications of type-II and type-I compactifications are stabilized [23, 26]. Another way to stabilize the moduli is to orbifold the theory by a symmetry that exists only for special values of the moduli [30]. The moduli are then fixed to take these special values. In this case, typically there are many additional massless scalar fields in the twisted sectors. These twisted moduli can in turn be made massive by including a shift in the orbifolding action. Using this mechanism for certain special asymmetric orbifolds, it is possible to construct models where *all* moduli except the dilaton are stabilized [31]–[34].

In this paper we investigate the relation between these various approaches. As we will see, in many respects compactifications with duality twists and internal fluxes are closely related to certain orbifolds with shifts.

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We review and develop the relevant aspects of compactification with duality twists in section 2 and illustrate the main points in section 3 with an example with SL(2) twists. We discuss the relation between duality twisting, fluxes, and orbifolds in section 4 and conclude in section 5 with some comments.

2. Compactification with duality twists

2.1 General Formalism

For simplicity, we consider twisted reduction on a circle but these results can be readily extended to more general toroidal compactifications.

Consider a $D+1$ dimensional supergravity (or theory of matter coupled to gravity) with a global symmetry G. An element g of the symmetry group acts on a generic field ψ as $\psi \to g[\psi]$. Consider now a dimensional reduction of the theory to D dimensions on a circle of radius R with a periodic coordinate $y \sim y + 2\pi R$. In the twisted reduction, the fields are not independent of the internal coordinate but are chosen to have a specific dependence on the circle coordinate y through the ansatz

$$\psi(x^\mu, y) = g(y)\, [\psi(x^\mu)] \tag{2.1}$$

for some y-dependent group element $g(y)$. An important restriction on $g(y)$ is that the reduced theory in D dimensions should be independent of y. This is achieved by choosing

$$g(y) = \exp\left(\frac{My}{2\pi R}\right) \tag{2.2}$$

for some Lie-algebra element M. The map $g(y)$ is not periodic around the circle, but has a *monodromy*

$$\mathcal{M}(g) = \exp M\,. \tag{2.3}$$

The Lie algebra element M generates a one-dimensional subgroup L of G.

It has been seen in explicit examples that Scherk-Schwarz reduction of a supergravity gives rise to a gauged supergravity; see e.g. [5, 7, 10, 11, 13]. It is easy to see that this must always be the case. Consider a field ψ in the $D+1$ dimensional theory that transforms in some representation of G as $\delta\psi = \epsilon\overline{M}\psi$ where ϵ is an infinitesimal parameter and $\overline{M}$ is the matrix representation of the element M. It is straightforward to show that on twisted dimensional reduction to D dimensions, the derivative of ψ is replaced by the gauge covariant derivative $\nabla\psi = d\psi + A\overline{M}\psi$, where A is the 1-form gauge potential arising from the Kaluza-Klein reduction of the metric on the circle. This follows from demanding general coordinate invariance under transformations of the form $y \to y + \delta y(x)$ where x are the coordinates of the noncompact D-dimensional spacetime. We thus obtain a gauged supergravity where L has become a local symmetry whose gauge field is the Kaluza-Klein vector potential. The gauged supergravity has fermion mass terms and modifications of the fermion supersymmetry transformations which are linear in the mass matrix M, and a scalar potential (discussed below) which is quadratic in M. If any other vector fields in the theory are singlets under G, then the gauge group is the one-dimensional group L (or

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strictly speaking the product of L with the gauge group for the other vector fields, which is abelian in most of the examples of interest here). However, if there are n other abelian gauge fields A^b $(a, b = 1, \ldots, n)$ transforming in some representation of G, $\delta A^a = \epsilon \overline{M}^a{}_b A^b$, then the gauge group is the semi-direct product of L with $\mathrm{U}(1)^n$ with generators t_y, t_a and structure constants $f_{ya}{}^b = -f_{ay}{}^b = \overline{M}^b{}_a$, where t_y is the generator corresponding to the Kaluza-Klein vector field and all other structure constants vanish.

The Scherk-Schwarz ansatz (2.1) breaks the global symmetry G down to the subgroup that commutes with $g(y)$. Acting with a general constant element h in G will change the twist to $hg(y)h^{-1}$ and would seem to give a new theory. However, this theory is related to the original one via the field redefinition $\psi \to h[\psi]$ for all fields ψ, so that the two choices of $g(y)$ in the same conjugacy class give equivalent reductions related by field-redefinitions [8].

The map $g(y)$ is a local section of a principal fiber bundle over the circle with fiber G and monodromy $\mathcal{M}(g)$ in G. Such a bundle is constructed from $I \times G$, where I is the interval $[0, 2\pi R]$, by gluing the ends of the interval together with a twist of the fibers by the monodromy $\mathcal{M}$. Two such bundles with monodromy in the same G-conjugacy class are equivalent.

In classical supergravity, any twist in G is allowed, but in M-theory, the twists must belong to the duality group $G(\mathbb{Z})$ and thus the inequivalent twisted reductions will be classified by the conjugacy classes of the discrete group $G(\mathbb{Z})$ [8]. Monodromies in $G(\mathbb{Z})$ related by G conjugation define theories with equivalent actions, but in general the action of G changes the charge lattice. For a fixed charge lattice, the equivalent classes of theories are defined by the classes of $G(\mathbb{Z})$ monodromy related by $G(\mathbb{Z})$ conjugation [8].

Note that in performing twisted reductions, it is not necessary that the potential have any critical points, or that the theory have a solution which is flat space or (anti-) de Sitter space in D dimensions. For example, in the twisted reduction of IIB supergravity in [2] the resulting $D = 9$ theory has a potential without critical points and so has no Minkowski or maximally symmetric vacua. However, it does have half-supersymmetric domain wall solutions, which can be lifted to solutions of the 10-dimensional IIB theory, as can any other solution of the $D = 9$ theory. This is a typical situation, and it is useful to discuss reduction in generality without specifying a D-dimensional solution.

Going around the circle many times generates twists that are powers of the monodromy $\mathcal{M}$. We will refer to the discrete abelian subgroup of $G(\mathbb{Z})$ generated by the monodromy $\mathcal{M}$ as the twist group of the bundle. If the order of the twist group is a finite integer n, then the n-fold cover of this fiber bundle is trivial because all twists can be completely undone around a larger circle. That is, with the ansatz (2.1) and (2.2) and twist group $\mathbb{Z}_n$, if the range of y is extended to run from 0 to $2\pi nR$, then the n-fold cover of the original circle is the circle with the identification $y \sim y + 2\pi nR$ and the monodromy for this covering circle is the identity, as $\mathcal{M}^n = \mathbb{1}$.

As we explain in section 2.3, the low energy effective action of the gauged supergravity in D dimensions is completely determined by the mass matrix M for a given monodromy $\mathcal{M}$. This leads to an apparent paradox. It is clear from eq. (2.3) that a given monodromy matrix can arise in general from infinitely many different mass matrices M [8]. As the

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bundle space is determined completely by the monodromy, different choices of M with the same $\mathcal{M}$ should give equivalent theories. On the other hand, as the mass matrix M appears explicitly in the gauged supergravity action, different choices of M would appear to give different theories. For example, in the case of trivial reduction with $\mathcal{M} = \mathbb{1}$, there are infinitely many mass matrices M satisfying $e^M = \mathbb{1}$, each of which would give a different supergravity action. We describe in the next subsection how this ambiguity is resolved.

2.2 An ambiguity

Consider the example of a complex scalar field ϕ reduced on a circle with coordinate y with the identification $y \sim y + 2\pi R$. For a trivial reduction, one has the mode expansion

$$\phi(x, y) = \sum_n e^{iny/R} \phi_n(x) \,, \tag{2.4}$$

giving an infinite set of fields $\phi_n(x)$ in the reduced theory with mass $m_n \propto n/R$, so that ϕ_0 is a massless field and the other modes are massive Kaluza-Klein modes. If the original theory is invariant under U(1) phase rotations $\phi \to e^{i\alpha}\phi$, one can include a U(1) twist in the reduction, so that the 1×1 mass matrix is $M = im/R$ for some real number m, with monodromy $\mathcal{M} = e^{2\pi i m}$. Then the twisted mode sum becomes

$$\phi(x, y) = \sum_n e^{i(n+m)y/R} \tilde{\phi}_n(x) \,, \tag{2.5}$$

so that the new modes $\tilde{\phi}_n(x)$ have mass $\tilde{m}_n \propto (n + m)/R$. Clearly, if m is an integer, then the two mode sums are equivalent, with $\tilde{\phi}_n = \phi_{n+m}$, and the full Kaluza-Klein spectra are the same, as one would expect from the fact that both reductions have monodromy matrix $\mathcal{M} = \mathbb{1}$. However, in the twisted case the mass matrix is non-trivial. This means that if one reduces and then truncates to the $n = 0$ sector, one is left with a single scalar field $\tilde{\phi}_0(x)$ with mass m/R, with different masses for different choices of integer m. In this way, one could truncate the Kaluza-Klein spectrum to any one of the massive modes $\phi_m = \tilde{\phi}_0$ instead of the usual choice ϕ_0. Similarly, two non-integral choices of mass $m = m_1, m = m_2$ which differ by an integer would give equivalent Kaluza-Klein spectra, but if one truncated to the $n = 0$ sector, one would obtain distinct truncations.

This applies more generally. The twisted compactifications are classified by the monodromy matrices, up to conjugation. Different choices of mass matrix which give equivalent monodromies will give equivalent Kaluza-Klein spectra, but can give distinct truncations to the 'zero-mode' sector (the analogue of the $n = 0$ sector in the example above whose only dependence on the extra coordinates comes from the twist). These different truncations will give different potentials as they depend on the mass matrix explicitly. However, in deriving low-energy effective physics, it is important to choose the truncation to the lightest fields. In the example above, the tower of Kaluza-Klein fields $\tilde{\phi}_n(x)$ have mass $\tilde{m}_n \propto (n + m)/R$ and one could truncate to a single scalar for any given value of n. However, the lightest scalar is for that value of n which minimizes $|m + n|$ and in deriving the effective low-energy physics, it is important to choose that value of n if one truncates, so that the effective theory describes the lightest states.

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2.3 The Scalar Potential

The moduli fields, which we generically denote by Φ, are not massless in the reduced theory in general and there is a nontrivial Scherk-Schwarz potential $V(\Phi)$ on the moduli space. It is straightforward to extend the analysis of Scherk and Schwarz [1] and later generalizations to obtain an explicit formula for this scalar potential in terms of the mass matrix M. For the case in which the scalars in $D+1$ dimensions take values in a coset G/K (typically G is a non-compact group with a maximal compact group K) they can be represented by a vielbein $\mathcal{V}(x) \in G$ transforming under rigid G transformations and local K transformations as $\mathcal{V} \to k(x)\mathcal{V}g$. Here we will restrict ourselves to the case in which $\mathcal{V}$ is a real matrix in a real representation of G; the generalization to complex representations is straightforward. The kinetic term is

$$L = -\frac{1}{2}\mathrm{Tr}[\mathcal{V}^{-1}D_m\mathcal{V}\mathcal{V}^{-1}D^m\mathcal{V}] \tag{2.6}$$

where D_m is a K-covariant derivative with K-connection given in terms of $\mathcal{V}$ and its derivative. In this formulation, the theory has a rigid G symmetry and a local K symmetry. The local K symmetry can be fixed to remove the unphysical degrees of freedom in $\mathcal{V}$. Let η be a constant K-invariant metric (for semi-simple K, it can be taken to be the Cartan-Killing metric, and for the standard case in which K is compact, a Lie algebra basis can be chosen so that $\eta = \mathbb{1}$). Then one can define the K-invariant field $\mathcal{H} = \mathcal{V}^t\eta\mathcal{V}$ transforming under G as $\mathcal{H} \to g^t\mathcal{H}g$, so that the kinetic term becomes

$$L = +\frac{1}{2}\mathrm{Tr}[\partial_m\mathcal{H}^{-1}\partial^m\mathcal{H}]\,. \tag{2.7}$$

It is straightforward to show that dimensional reduction on a circle with a twist determined by the mass matrix M yields a potential in D dimensions given by

$$V(\Phi) = e^{a\phi}\mathrm{Tr}[M^2 + M^t\mathcal{H}(\Phi)M\mathcal{H}^{-1}(\Phi)]\,, \tag{2.8}$$

where e^ϕ is the modulus corresponding to the radius of the circle and $a = 6/(D-1)(D-2)$. The potential arises from the y-derivatives in eq. (2.7) with the Scherk-Schwarz ansatz $\mathcal{H}(\Phi(x), y) = \mathcal{M}^t(y)\mathcal{H}(\Phi(x))\mathcal{M}(y)$ with $\mathcal{M}(y) = \exp\frac{My}{2\pi R}$. The matrix M has dimensions of mass and introduces mass parameters into the theory. This generalizes the results of [1, 9, 10].

One immediate question is whether this potential has any stable minima and which moduli acquire mass at these minima. In terms of $\tilde{M} = \mathcal{V}M\mathcal{V}^{-1}$, the potential becomes

$$V(\Phi) = e^{a\phi}\mathrm{Tr}[\tilde{M}^2 + \tilde{M}^t\eta\tilde{M}\eta^{-1}]\,. \tag{2.9}$$

For a given mass matrix M, the potential depends on the moduli Φ that parametrize the coset through the matrix $\tilde{M}(\Phi)$. The dependence on ϕ is only through the exponential factor, so the potential will be stationary with respect to variations of ϕ only if $V(\Phi) = 0$, which requires either $a\phi = -\infty$, or $\tilde{M} = \tilde{M}_0$ with

$$\mathrm{Tr}[\tilde{M}_0(\tilde{M}_0 + \eta^{-1}\tilde{M}_0^t\eta)] = 0\,. \tag{2.10}$$

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Let us now restrict to the case in which K is compact and η is the identity matrix (e.g. $G = \mathrm{SL}(N)$ and $K = \mathrm{SO}(N)$). Then the potential can be rewritten as

$$V(\Phi) = \frac{1}{2} e^{a\phi} \mathrm{Tr}(Y^2) \tag{2.11}$$

where Y is the real symmetric matrix, $Y \equiv [\tilde{M} + \tilde{M}^t]$. The potential is then manifestly positive, $V(\Phi) \geq 0$ because Y is diagonalizable with real eigenvalues, so that $\mathrm{Tr}(Y^2)$ is the sum of the squares of the eigenvalues. It is clear that the potential will vanish at a point $\Phi = \Phi_0$ in the moduli space if and only if Y vanishes at that point. At such a point Φ_0 at which $Y = 0$, $\tilde{M}(\Phi_0)$ equals a rotation generator $\tilde{M}_0$ with $\tilde{M}_0 = -\tilde{M}_0^t$. Moreover, from the positivity of the potential, the point Φ_0 is a global minimum that is stable or at least marginally stable. Given such an antisymmetric $\tilde{M}_0$, the relation $\tilde{M}_0 = \mathcal{V}_0 M \mathcal{V}_0^{-1}$ determines the corresponding value $\mathcal{V}_0$ of the vielbein $\mathcal{V}$ at the point $\Phi = \Phi_0$. To summarize, the only critical points of the potential for finite ϕ are the stable minima where the potential vanishes and where $\tilde{M}(\Phi_0)$ is a rotation generator.

We now derive some general properties of the critical points of this potential which will play a vital role in understanding the relation between twisted reductions and orbifolds. We will show that the critical points (or submanifolds) are fixed under the action of the twist group. The relevant mathematics will be discussed further in section 4.4. Consider then the case in which the mass matrix is G-conjugate to a rotation generator r, $r = -r^t$, so that

$$M = S^{-1} r S \tag{2.12}$$

for some constant $S \in G$. Then the monodromy $\mathcal{M} = e^M$ is conjugate to a rotation matrix $R = e^r$ satisfying $R^t R = \mathbb{1}$,

$$\mathcal{M} = S^{-1} R S\,. \tag{2.13}$$

The potential now will have a global minimum at the point Φ_0 in moduli space such that $\mathcal{V}(\Phi_0) = S$ because at that point $\tilde{M}_0 = r$ and so $Y(\Phi_0) = 0$. At this point, the coset metric takes the value $\mathcal{H}_0 = S^t S$. This is invariant under the action of the twist group, $\mathcal{H}_0 \to \mathcal{H}_0' \equiv \mathcal{M}^t \mathcal{H}_0 \mathcal{M} = \mathcal{H}_0$, as is easily seen using (2.13) and $R^t R = \mathbb{1}$. Thus, such a critical point is a fixed point under the action of the twist group generated by $\mathcal{M}$.

There is a natural action of G on the theory, inherited from the structure of the $D+1$ dimensional theory, but it is not a symmetry in D dimensions, as the mass terms and potential are not invariant under G (although they are preserved by a subgroup). Acting with G is a field redefinition, and there are two situations to consider. First, if the $D+1$ dimensional theory is a field theory with a global G symmetry (e.g. a classical supergravity), then the field redefinition from acting with G takes the D-dimensional theory to an equivalent theory, written in terms of different variables. The second case is that in which the $D+1$ dimensional theory has only a $G(\mathbb{Z})$ symmetry (as in string theory or M-theory compactifications, or in a classical Kaluza-Klein reduction on $\mathbf{T}^n$ where the massive Kaluza-Klein modes break the low-energy $\mathrm{SL}(n,\mathbb{R})$ to $\mathrm{SL}(n,\mathbb{Z})$). If there is a charge lattice acted on by G and preserved by the subgroup $G(\mathbb{Z})$, then for a fixed charge lattice, only field redefinitions from the action of $G(\mathbb{Z})$ will lead to equivalent theories.

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Since G acts transitively on the coset, any point on the coset Φ_0 can be moved to any other point Φ_0' by right multiplication of the vielbein by some element $U \in G$, $\mathcal{V}(\Phi_0) \to \mathcal{V}(\Phi_0') = \mathcal{V}(\Phi_0)U$. Under this action, the twist $\mathcal{M}$ will go to $\mathcal{M}' = U^{-1}\mathcal{M}U$, changing the potential to a new one. If Φ_0 was a critical point of the original potential, then Φ_0' is a critical point of the new one. In the first situation in which G is a symmetry in $D+1$ dimensions, this action of G is a field redefinition and leads to an equivalent theory and by acting with G, any given critical point Φ_0 can be moved to any desired point in moduli space Φ_0'. In the second situation in which the original theory only has a $G(\mathbb{Z})$ symmetry, acting with G in general takes the theory to an inequivalent one, but acting with $G(\mathbb{Z})$ leads to an equivalent theory. Thus acting with G can move a critical point to any desired point in moduli space, but in general changes the theory. Acting with $G(\mathbb{Z})$ will take the theory to a physically equivalent one, and change the monodromy to another representative of the same $G(\mathbb{Z})$ conjugacy class. The $G(\mathbb{Z})$ action can be used to move any critical point to one in a fundamental domain $G(\mathbb{Z})\backslash G/K$ of the moduli space. However, then acting with G to move it to another point in the same fundamental domain would lead to an inequivalent theory.

The distinction between these two situations will be important later when we discuss orbifolds in section 4. Different points in the moduli space where different orbifold theories are possible can be moved to each other by G transformations and would appear to be equivalent in the naive low-energy analysis unless we correctly incorporate the integrality of charges as above by allowing only $G(\mathbb{Z})$ transformations.

3. Examples with SL(2) Twists

We now illustrate the main ingredients of this construction by means of an example of a standard reduction on $\mathbf{T}^2$ followed by a twisted reduction on $\mathbf{S}^1$. Reducing first on the $\mathbf{T}^2$ gives a theory whose symmetries include the mapping class group $\mathrm{SL}(2,\mathbb{Z})$ of the torus. One can then reduce further on the circle with a twist that belongs to this $\mathrm{SL}(2,\mathbb{Z})$. This example will also prepare the background for establishing the connection with orbifolds, and is closely related to the IIB compactifications considered in [8, 9, 10, 35, 11, 13].

3.1 Pure Gravity

Consider first a theory of pure gravity with Einstein-Hilbert action in $D+3$ dimensions. Dimensionally reducing on $\mathbf{T}^2$ gives a theory in $D+1$ dimensions whose massless spectrum contains the graviton, two Kaluza-Klein gauge bosons and three scalar fields coming from the moduli of the torus. The area of the torus e^{ψ} parametrizes $\mathbb{R}^+$ and the complex structure τ of the torus parametrizes $\mathrm{SL}(2,\mathbb{Z})\backslash \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$. The $\mathrm{SL}(2,\mathbb{Z})$ is the group of large diffeomorphisms of the torus and is a discrete gauge symmetry.

The truncated massless theory in $D+1$ dimensions now has $\mathrm{SL}(2,\mathbb{R})$ global symmetry and we can consider the reduction on a further circle to D dimensions with an $\mathrm{SL}(2,\mathbb{R})$ twist. There are three distinct twisted reductions corresponding to the three distinct $\mathrm{SL}(2,\mathbb{R})$ conjugacy classes [8]. These are the hyperbolic, elliptic and parabolic $\mathrm{SL}(2,\mathbb{R})$

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conjugacy classes, represented by the monodromy matrices

$$\mathcal{M}_h = \begin{pmatrix} e^m & 0 \\ 0 & e^{-m} \end{pmatrix}, \qquad \mathcal{M}_e = \begin{pmatrix} \cos m & \sin m \\ -\sin m & \cos m \end{pmatrix}, \qquad \mathcal{M}_p = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \tag{3.1}$$

respectively, generated by the matrices

$$M_h = \begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix}, \qquad M_e = \begin{pmatrix} 0 & m \\ -m & 0 \end{pmatrix}, \qquad M_p = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \tag{3.2}$$

and each class is specified by a single coupling constant or mass parameter m.

For each of these theories the Scherk-Schwarz potential (2.8) takes a simple form. The scalars $\psi, \tau = \tau_1 + i\tau_2$ take values in $GL(2,\mathbb{R})/\,\mathrm{SO}(2)$ and can be represented by the $GL(2,\mathbb{R})$ matrix $\mathcal{V}$ with a local SO(2) invariance removing one of the four degrees of freedom of $\mathcal{V}$. Then $\mathcal{H} = \mathcal{V}^t\mathcal{V}$ can be given in terms of ψ, τ as $\mathcal{H} = e^{\psi}H(\tau)$ where

$$H(\tau) \equiv \frac{1}{\tau_2}\begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix} \tag{3.3}$$

and the potential is given by

$$V(\tau) = e^{a\phi}\mathrm{Tr}[M^2 + M^t H(\tau) M H^{-1}(\tau)]\,. \tag{3.4}$$

Note that the potential is independent of ψ. For the elliptic twisting with monodromy $\mathcal{M}_e$, the potential has a minimum at $\tau = i$ giving a Minkowski vacuum. For the parabolic case, the potential is proportional to $m^2 e^{a\phi + b\Phi}$ where $\tau_2 = e^{-\Phi}$ and and b is a constant, and so the only critical points are when $a\phi + b\Phi = -\infty$. For finite ϕ, this corresponds to $\tau = i\infty$, representing a degenerate torus. The hyperbolic case has no critical points on the upper half plane.

The $\mathrm{SL}(2,\mathbb{R})$ global symmetry of the massless reduction is broken down to an $\mathrm{SL}(2,\mathbb{Z})$ subgroup if the massive Kaluza-Klein states are kept. For the reduction of the full Kaluza-Klein theory including the massive states, therefore, the monodromy must belong to $\mathrm{SL}(2,\mathbb{Z})$. The $\mathrm{SL}(2,\mathbb{Z})$ conjugacy classes have been analyzed in [36, 37]. For any conjugacy class $\mathcal{M}$, $-\mathcal{M}$ and $\pm\mathcal{M}^{-1}$ also represent conjugacy classes, so for each $\mathcal{M}$ in the following list, there are also conjugacy classes $-\mathcal{M}$ and $\pm\mathcal{M}^{-1}$.

Apart from the trivial class $\mathcal{M} = \mathbb{1}$, there are four conjugacy classes that generate twist groups of finite order

$$\mathcal{M}_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathcal{M}_3 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \quad \mathcal{M}_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathcal{M}_6 = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.5}$$

The matrices $\mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4, \mathcal{M}_6$ respectively generate $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$ subgroups of $\mathrm{SL}(2,\mathbb{Z})$ and the subscript gives the order of the subgroup. The monodromies $\mathcal{M}_3, \mathcal{M}_4, \mathcal{M}_6$ are all in the elliptic conjugacy class of $\mathrm{SL}(2,\mathbb{R})$ with $|Tr(\mathcal{M})| < 2$.

The monodromies in the parabolic and hyperbolic conjugacy classes all generate twist groups of infinite order. There are an infinite number of parabolic $\mathrm{SL}(2,\mathbb{Z})$ conjugacy classes with $Tr(\mathcal{M}) = 2$, represented by T^n:

$$\mathcal{M}_{T_n} = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \tag{3.6}$$

with a distinct conjugacy class for each integer n.

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There are an infinite number of hyperbolic $\mathrm{SL}(2,\mathbb{Z})$ conjugacy classes with $|Tr(\mathcal{M})| > 2$, represented by

$$\mathcal{M}_{H_n} = \begin{pmatrix} n & 1 \\ -1 & 0 \end{pmatrix}, \tag{3.7}$$

for integers n with $|n| \geq 3$, together with sporadic monodromies $\mathcal{M}(t)$ of trace t

$$\begin{aligned} \mathcal{M}(8) &= \begin{pmatrix} 1 & 2 \\ 3 & 7 \end{pmatrix}, \qquad \mathcal{M}(10) = \begin{pmatrix} 1 & 4 \\ 2 & 9 \end{pmatrix}, \qquad \mathcal{M}(12) = \begin{pmatrix} 1 & 2 \\ 5 & 11 \end{pmatrix} \\ \mathcal{M}(13) &= \begin{pmatrix} 2 & 3 \\ 7 & 11 \end{pmatrix}, \qquad \mathcal{M}(14) = \begin{pmatrix} 1 & 2 \\ 6 & 13 \end{pmatrix}, \ldots \end{aligned} \tag{3.8}$$

and this gives the complete list of sporadic classes for $3 \leq t \leq 15$.

The mass matrices corresponding to the monodromies (3.5) and (3.6) are given by

$$\begin{aligned} M_2 &= \pi A^{-1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} A, \qquad M_3 = \frac{2\pi}{3\sqrt{3}} \begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix}, \qquad M_4 = \frac{\pi}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\ M_6 &= \frac{\pi}{3\sqrt{3}} \begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix}, \qquad M_{T^n} = \begin{pmatrix} 0 & n \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{3.9}$$

where A is an arbitrary $\mathrm{SL}(2,\mathbb{R})$ matrix.

The ambiguity discussed in section section 2.2 arises here from the infinitely many solutions of the equation $e^M = \mathbb{1}$ given by $M = 2\pi\,(0 \quad n - n \quad 0)$. This ambiguity does not affect the full physical spectrum and in (3.9) we have chosen, for each monodromy, a simple representative for the mass matrix from the infinite number of possible choices. Note that after accounting for this ambiguity, the mass matrices for the monodromies $\mathcal{M}_3, \mathcal{M}_4, \mathcal{M}_6$ are uniquely determined but there are still an infinite number of mass matrices M_2, characterized by the arbitrary matrix A, that all give rise to the same monodromy $\mathcal{M}_2$. Note that changing A is an $\mathrm{SL}(2,\mathbb{R})$ conjugation and so a field redefinition in the truncated theory in which the Kaluza-Klein modes are absent and the $D+1$ dimensional theory has an $\mathrm{SL}(2,\mathbb{R})$ symmetry, but for the full theory it changes the theory unless it is an $\mathrm{SL}(2,\mathbb{Z})$ conjugation. We shall return to the role of A in our discussion of orbifolds. Each of the mass matrices (3.9) is $\mathrm{SL}(2,\mathbb{R})$-conjugate to the mass matrix M_e in (3.2), $M_n = U^{-1} M_e U$ and so the corresponding potentials each have a unique critical point at which $V = 0$, and this is located at the image of $\tau = i$ under the action of the $\mathrm{SL}(2,\mathbb{R})$ transformation U.

3.2 Bosonic String

Consider next the bosonic string compactified on $\mathbf{T}^2$. In addition to the metric, we now also have a dilaton and an antisymmetric tensor among the massless fields. The global symmetry group is $G = O(2,2)$ and for fixed value of the dilaton, the moduli space of these compactifications is given by the Narain coset $O(2,2;\mathbb{Z})\backslash O(2,2)/O(2) \times O(2)$.

A convenient parametrization of this space is in terms of the complex structure modulus τ and the complexified Kähler modulus σ. The real part of σ is the area of the torus and the imaginary part is the value of the 2-form field B_{mn} on the torus. The moduli space for complex structures is $\mathrm{SL}(2,\mathbb{Z})\backslash \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ as before and the Kähler modulus

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parametrizes an identical space. The total moduli space is thus

$$[\mathrm{SL}(2,\mathbb{Z})\backslash\,\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}\,(2)\times\mathrm{SL}(2,\mathbb{Z})\backslash\,\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SO}\,(2)]/\mathbb{Z}_2\,. \tag{3.10}$$

The additional $\mathbb{Z}_2$ comes from the "parity" element of $O(2,2,\mathbb{Z})$ with determinant -1. This element changes the sign of one of the left-moving coordinates of the torus and hence corresponds to T-duality along that coordinate; it exchanges τ and σ and interchanges the two $\mathrm{SL}(2,\mathbb{Z})$ factors (see, for example, [38]).

We can now reduce the theory further on a circle with a duality twist given by a conjugacy class of $G(\mathbb{Z}) = [(\mathrm{SL}(2,\mathbb{Z})_\tau \times \mathrm{SL}(2,\mathbb{Z})_\sigma] \rtimes \mathbb{Z}_2$. The subscripts are added to denote that $\mathrm{SL}(2,\mathbb{Z})_\tau$ and $\mathrm{SL}(2,\mathbb{Z})_\sigma$ act on τ and σ respectively. The twists that belong to the $\mathrm{SL}(2,\mathbb{Z})_\tau$ factor have already been discussed in the previous subsection; there are distinct theories corresponding to each of the conjugacy classes of $\mathrm{SL}(2,\mathbb{Z})$. The twists by $\mathrm{SL}(2,\mathbb{Z})_\sigma$ are nongeometric but are conjugate by the $\mathbb{Z}_2$ T-duality element to $\mathrm{SL}(2,\mathbb{Z})_\tau$ and lead to equivalent theories. Twisting simultaneously by elements of the two $\mathrm{SL}(2)$ factors with a mass matrix

$$M = (M_\sigma \otimes \mathbb{1}) \oplus (\mathbb{1} \otimes M_\tau) \tag{3.11}$$

where M_σ and M_τ are mass matrices of $\mathrm{SL}(2)_\sigma$ and $\mathrm{SL}(2)_\tau$ twists respectively, results in new theories. As we discuss in section 4.1, these new theories are related to asymmetric orbifolds.

3.3 Supergravity

For a supergravity with a global symmetry G and local symmetry K, with scalars in G/K parametrized by $\mathcal{V}$, the fermions are inert under G but transform under K. In a physical gauge in which the K symmetry is fixed, a G transformation is accompanied by a compensating K transformation which acts on the fermions. Given the low energy action for the massless bosons, the effective action for the fermions is determined by supersymmetry. Corresponding to the nontrivial scalar potential (2.8), the fermions acquire moduli-dependent mass terms that are linear in the mass matrix M, and the supersymmetry transformations of the fermions are modified by terms linear in M.

Consider the Scherk-Schwarz reduction from $D+1$ to D dimensions on a circle, in the formalism in which the local K symmetry is not fixed. For the bosonic sector, the reduction is specified by the choice of a twist in G. In the fermionic sector, there is a choice of spin structure for the fermions on the circle (i.e. the possibility of including a twist by $(-1)^F$). The fermions can be decomposed into K representations, and in principle it is possible to choose a different spin structure for each K representation. In addition, there is the possibility of accompanying this by a twist in K.

Alternatively, one can first choose a physical gauge eliminating the local K symmetry, and then reduce with a twist in G (which acts on fermions through the compensating transformation) and a choice of spin structure for each K representation. In the cases that we have discussed so far, the symmetries include a rigid $\mathrm{SL}(2,\mathbb{R}) \subseteq G$ symmetry and a local $\mathrm{U}(1) \subseteq K$ in $D+1$ dimensions. In this case, if we fix the K symmetry completely by

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choosing physical gauge, the $\mathrm{SL}(2,\mathbb{R})$ transformation represented by the matrix

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{3.12}$$

will act on a fermion λ of U(1) charge q by the compensating U(1) transformation

$$\lambda \to \left(\frac{c\bar{\tau}+d}{c\tau+d}\right)^{q/4} \lambda\,. \tag{3.13}$$

Here we restrict ourselves to the case in which we twist only by the global group G and the spin structure is periodic for all fermions. This gives reductions specified by a mass matrix M which reduce to the standard reduction when $M=0$.

In the standard reduction on $\mathbf{T}^2$ followed by a twisted reduction on $\mathbf{S}^1$ that we have considered above, all gravitini become massive at the minima of the scalar potential and the supersymmetry is completely broken. This can be checked directly, and will become apparent once we make the connection with orbifolds. In the orbifold description, the gravitini have nontrivial transformations under the twist groups and are thus projected out, so that there are no massless gravitini in the spectrum and supersymmetry is completely broken. It is straightforward, however, to construct models with supersymmetric minima by compactifying on higher dimensional tori; we will discuss a simple example in section section 4.2.

3.4 Superstrings

For the heterotic string on $\mathbf{T}^2$, there are additional gauge fields and extra moduli from the Wilson lines. The Narain moduli space is now $O(2,18;\mathbb{Z})\backslash O(2,18)/O(2)\times O(18)$. On the submanifold of this moduli space where all Wilson lines are turned off, the duality symmetry is again $[(\mathrm{SL}(2,\mathbb{Z})_\tau \times \mathrm{SL}(2,\mathbb{Z})_\sigma] \rtimes \mathbb{Z}_2$. In this special case, the analysis is similar to that for the bosonic string. More general reductions twisted by conjugacy classes of the full duality group $O(2,18;\mathbb{Z})$ are quite interesting and are related to heterotic compactifications with various magnetic fluxes turned on, as will be discussed elsewhere.

For the type-IIA superstring on $\mathbf{T}^2$, the U-duality group is $\mathrm{SL}(3,\mathbb{Z})\times\mathrm{SL}(2,\mathbb{Z})$. The SL(3) is a symmetry of the supergravity action and contains $\mathrm{SL}(2)_\tau$, while the SL(2) factor is only a symmetry of the supergravity equations of motion and is the $\mathrm{SL}(2)_\sigma$ factor considered above. The perturbative T-duality symmetry is $[(\mathrm{SL}(2,\mathbb{Z})_\tau \times \mathrm{SL}(2,\mathbb{Z})_\sigma]$. Note that the $\mathbb{Z}_2$ element corresponding to T-duality along one leg of the torus is no longer a symmetry because it interchanges type-IIA with type-IIB. The type-IIB superstring compactified on $\mathbf{T}^2$ gives the same $D=8$ theory, but now for IIB it is $\mathrm{SL}(2)_\sigma$ that is contained in SL(3), while the SL(2) factor that is only a symmetry of the equations of motion is the geometric symmetry $\mathrm{SL}(2)_\tau$. Whereas in the heterotic or bosonic case, twisting by $\mathrm{SL}(2)_\tau$ or $\mathrm{SL}(2)_\sigma$ gave equivalent theories related by T-duality, in the type-II case they give rise to two distinct SL(2) twistings. In the first, the IIA theory is twisted by $\mathrm{SL}(2)_\tau$, and this is T-dual to twisting type-IIB by $\mathrm{SL}(2)_\sigma$. This results in a theory similar to the bosonic and the heterotic cases. In the second, the IIA theory is twisted by $\mathrm{SL}(2)_\sigma$, and this is T-dual to twisting type-IIB by $\mathrm{SL}(2)_\tau$. In this case, the twist is by a symmetry that acts via duality and is only a symmetry of the equations of motion, not of the action. This results in some novel features, which will be analyzed in [39].

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For type-II strings there are other more general possibilities when the twisting is non-perturbative and the monodromy is an arbitrary element of $\mathrm{SL}(3,\mathbb{Z}) \times \mathrm{SL}(2,\mathbb{Z})$. For example, the type-IIB string in $D = 10$ has a nonperturbative $\mathrm{SL}(2,\mathbb{Z})_\lambda$ symmetry that acts on the dilaton-axion field λ. After reducing on $\mathbf{T}^2$ this $\mathrm{SL}(2,\mathbb{Z})_\lambda$ becomes a subgroup of $\mathrm{SL}(3,\mathbb{Z})$ and is conjugate to the perturbative $\mathrm{SL}(2,\mathbb{Z})_\sigma$ discussed above. Therefore, the $\mathrm{SL}(2,\mathbb{Z})_\lambda$ twists are dual to the $\mathrm{SL}(2,\mathbb{Z})_\sigma$ twists. Even though the group theoretic considerations are identical in the two cases, the realization in terms of perturbative string modes will be quite different. For example, twists that correspond to turning on NS-NS fluxes will be conjugate to twists that correspond to turning on R-R fluxes.

Note that the $D = 7$ theory obtained by twisting with an element of the $\mathrm{SL}(2,\mathbb{Z})_\lambda$ can also be obtained by first reducing the IIB theory on a circle with an $\mathrm{SL}(2,\mathbb{Z})_\lambda$ twist $\mathcal{M}$ to $D = 9$, and then performing a standard reduction on $\mathbf{T}^2$. Thus, the $D = 7$ theories obtained by twisting with $\mathrm{SL}(2,\mathbb{Z})_\lambda$ are precisely the $\mathbf{T}^2$ reductions of the $D = 9$ theories of [9, 10, 8, 11, 13] and have a very similar structure. The $D = 9$ theory can be thought of as F-theory compactified on a $\mathbf{T}^2$ bundle over $\mathbf{S}^1$ with monodromy $\mathcal{M}$ [8].

4. Orbifolds, duality twists, and fluxes

Given a theory with a discrete symmetry $\mathbf{X}$, its orbifold is obtained by gauging the symmetry. The Hilbert space of the orbifold consists of states of the original theory that are invariant under $\mathbf{X}$, together with new twisted string states that are closed up to a nontrivial $\mathbf{X}$ transformation. We will be interested in orbifolds of strings compactified on $\mathbf{T}^2 \times \mathbf{S}^1$. For special values of the torus modulus, the torus will be invariant under a discrete $\mathbb{Z}_n$ symmetry of finite order $n = 2, 3, 4$ or 6. For such a torus, the orbifold group $\mathbf{X} = \mathbb{Z}_n$ relevant for our purpose is generated by a $\mathbb{Z}_n$ generator of the torus symmetry group accompanied by an order n shift along the circle.

4.1 Bosonic string

Let us first consider orbifolds of the bosonic string where the discrete rotation is geometric and acts symmetrically on the left-moving and right-moving coordinates of the torus. To see what geometric rotations are allowed, let z be the complex coordinate of $\mathbf{T}^2$ with the identifications $z \sim z + 1 \sim z + \tau$, where τ is the complex structure modulus of the torus. For what follows, the Kähler modulus can be arbitrary so the over-all scale of the torus is not important. Associated with the torus is a lattice of points in the complex plane, $\{z = m + n\tau\}$, for arbitrary integers m and n. Now, a rotation in the complex plane becomes a symmetry of the torus only if it is a symmetry of the lattice. A $\mathbb{Z}_2$ rotation through π that takes z to $-z$ is a symmetry of all lattices. Additional symmetries are possible for special lattices (i.e. for special values of τ) given by the crystallographic classification [40]. A square lattice with $\tau = i$ has an enhanced $\mathbb{Z}_4$ symmetry generated by the rotations $z \to e^{i\pi/2} z$ and a hexagonal lattice with $\tau = e^{2\pi i/3}$ has an enhanced $\mathbb{Z}_6$ symmetry generated by $z \to e^{i\pi/3} z$ with a $\mathbb{Z}_3$ subgroup generated by $z \to e^{2i\pi/3} z$. The only possible discrete rotation symmetries of the torus are $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$.

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The orbifold action for our purposes will be one of these $\mathbb{Z}_n$ rotations of a torus at a special value of the modulus with a simultaneous order n shift along the circle of radius nR for $n = 2, 3, 4, 6$. Note that the list of allowed orbifold rotations is in one–to–one correspondence with the list of twist groups generated by the monodromies $\mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4, \mathcal{M}_6$ that we encountered earlier in a rather different context. We now explain the relation between the orbifolds and the twisted reductions.

It is clear that all of the above orbifolds can be viewed as twisted reductions. The group $\mathrm{SL}(2,\mathbb{Z})$ of large diffeomorphisms of $\mathbf{T}^2$ has a natural action on the lattice defining the torus and the $\mathbb{Z}_n$ symmetry of a special lattice is a subgroup of $\mathrm{SL}(2,\mathbb{Z})$ that leaves the lattice invariant. Conjugation by $\mathrm{SL}(2,\mathbb{Z})$ gives a physically equivalent rotation and thus again there is a dependence only on conjugacy classes. If the circle has radius r and coordinate $y \sim y + 2\pi r$, then the orbifolded theory is identified under the action of a $\mathbb{Z}_n$ rotation accompanied by a shift $y \to y + 2\pi r/n$. This is equivalent to the twisted reduction on a circle of radius $R = r/n$ with a twist by the $\mathbb{Z}_n$ generator. Since the orbifold satisfies the string equations of motion with vanishing ground state energy at tree level, the Scherk-Schwarz potential must have a stable (or marginally stable) minimum with zero energy at this point.

The converse is more interesting and less obvious. Compactification with a duality twist is more general than the orbifold construction in certain respects because it can be carried out without restricting the moduli to special values and the moduli can have nontrivial variation along the circle and in the D-dimensional spacetime. Moreover, we can twist by any monodromy, giving distinct theories for each of the infinite number of conjugacy classes listed in section 3. The orbifold, on the other hand, is possible only for special values of the moduli where the lattice admits a symmetry and the class of allowed orbifold rotations is finite. As we now discuss, the connection between the two is provided by the Scherk-Schwarz potential. The minima of the potential occur precisely at the fixed points in the coset space $\mathrm{SL}(2)/\mathrm{SO}(2)$ under the action of the twist group, and these are precisely the points in moduli space where orbifolding is possible.

Consider first the parabolic and the hyperbolic conjugacy classes of $\mathrm{SL}(2,\mathbb{Z})$. Monodromies in these conjugacy classes generate twist groups of infinite order and have no fixed points on the upper half plane with τ_2 strictly positive and finite. As discussed in section 3, the Scherk-Schwarz potential has no stable minima with τ_2 strictly positive and finite in these cases, consistent with the fact that there is no standard orbifold formulation in this situation.

Monodromies in the elliptic conjugacy classes of $\mathrm{SL}(2,\mathbb{Z})$ generate twists of finite order. As they are $\mathrm{SL}(2,\mathbb{R})$-conjugate to a rotation, they must have a fixed point. In fact, it follows from a theorem given in section 4.4 that any finite order subgroup of $G(\mathbb{Z})$ always has a fixed point on G/K for any non-compact semi-simple G with K its maximal compact subgroup. Moreover, together with the discussion in section 2.3 this implies that the Scherk-Schwarz potential for a given elliptic monodromy has a stable minimum precisely at this fixed point. We now check these facts by hand for the simple case of $\mathrm{SL}(2)$ by explicitly finding the minima of the potential for the mass matrices given by (3.9).

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When $G = \mathrm{SL}(2)$, the vielbein can always be written in the physical gauge as an upper triangular matrix with the parametrization

$$\mathcal{V}(\tau) = \frac{1}{\sqrt{\tau_2}} \begin{pmatrix} 1 & \tau_1 \\ 0 & \tau_2 \end{pmatrix} \tag{4.1}$$

so that the metric $\mathcal{H} = \mathcal{V}^t \mathcal{V}$ takes the canonical form (3.3). (That this can always be done is seen most easily by using the Iwasawa decomposition of a general SL(2) matrix as a product $k\mathcal{V}$ where k is an SO(2) matrix and $\mathcal{V}$ is an upper triangular matrix and then fixing the physical gauge to gauge away k.) In this parametrization, given an arbitrary mass matrix $M = (\,-d \quad bc \quad d\,)$ in the Lie algebra of $\mathrm{SL}(2, \mathbb{R})$, the matrix $\tilde{M} = \mathcal{V} M \mathcal{V}^{-1}$ is given by

$$\tilde{M} = \frac{1}{\tau_2} \begin{pmatrix} 1 & \tau_1 \\ 0 & \tau_2 \end{pmatrix} \begin{pmatrix} -d & b \\ c & d \end{pmatrix} \begin{pmatrix} \tau_2 & -\tau_1 \\ 0 & 1 \end{pmatrix} = \frac{1}{\tau_2} \begin{pmatrix} -d\tau_2 + c\tau_1\tau_2 & d\tau_1 + b - c{\tau_1}^2 + d\tau_1 \\ c{\tau_2}^2 & -c\tau_1\tau_2 + d\tau_2 \end{pmatrix}. \tag{4.2}$$

Now we have seen in section 2.3, the potential can be written in the form

$$V(\tau) = \frac{1}{2} e^{a\phi} \mathrm{Tr}(Y^2) \tag{4.3}$$

where Y is a real symmetric matrix, $Y \equiv [\tilde{M} + \tilde{M}^t]$. Therefore, for a given mass matrix M, a minimum occurs precisely for those values of τ for which the corresponding Y matrix vanishes. The Y matrices corresponding to the four mass matrices in (3.9) for the elliptic conjugacy classes are given by

$$\begin{aligned} Y_2 &= \begin{pmatrix} -d\tau_2 + c\tau_1\tau_2 & d\tau_1 + b - c{\tau_1}^2 + c{\tau_2}^2 + d\tau_1 \\ d\tau_1 + b - c{\tau_1}^2 + c{\tau_2}^2 + d\tau_1 & -c\tau_1\tau_2 + d\tau_2 \end{pmatrix} \\ Y_3 &= \frac{4\pi}{3\sqrt{3}\tau_2} \begin{pmatrix} \tau_2 - 2\tau_1\tau_2 & 1 + {\tau_1}^2 - {\tau_2}^2 - \tau_1 \\ 1 + {\tau_1}^2 - {\tau_2}^2 - \tau_1 & -\tau_2 + 2\tau_1\tau_2 \end{pmatrix} = 2Y_6 \\ Y_4 &= \frac{\pi}{2\tau_2} \begin{pmatrix} -2\tau_1\tau_2 & 1 + {\tau_1}^2 - {\tau_2}^2 \\ 1 + {\tau_1}^2 - {\tau_2}^2 & 2\tau_1\tau_2 \end{pmatrix}. \end{aligned} \tag{4.4}$$

Note that in the matrix Y_2, the three real numbers b, c, d are subject to the constraint $d^2 + bc = -\pi^2$ and thus it depends effectively on only two parameters. This follows from the fact that the mass matrix M_2 in (3.9) depends on an arbitrary $\mathrm{SL}(2, \mathbb{R})$ matrix A and is an arbitrary trace-less matrix whose determinant equals π^2.

Now, the minima of the potential can be readily found. The matrices Y_3 and Y_6 vanish only at $\tau = \exp(\pi i/3)$ and thus for twists by the monodromies $\mathcal{M}_3$ and $\mathcal{M}_6$, the minimum of the potential (4.3) occurs precisely at points where a $\mathbb{Z}_3$ and $\mathbb{Z}_6$ orbifold action is possible. Similarly the matrix Y_4 vanishes only at $\tau = i$ and thus for the monodromy $\mathcal{M}_4$ the potential has a minimum precisely where a $\mathbb{Z}_4$ orbifold is possible.

For the conjugacy class $\mathcal{M}_2$, the position at which the matrix Y_2 vanishes depends on the choice of the numbers b, c, d in (4.4), corresponding to the choice of the $\mathrm{SL}(2, \mathbb{R})$ matrix A in (3.9). Choosing $A = 1$, $d = 0, b = -c$, Y_2 vanishes at $\tau = i$. Now conjugating with $U \in \mathrm{SL}(2, \mathbb{R})$ gives $A = U$ and can be used to move the point at which Y_2 vanishes to any desired point in moduli space. Changing A in this way changes the compactified theory

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unless $A \in \mathrm{SL}(2,\mathbb{Z})$, and this $\mathrm{SL}(2,\mathbb{Z})$ redundancy can be used to move the critical point into a fundamental domain. This freedom is consistent with the fact that a $\mathbb{Z}_2$ orbifold is possible for all values of τ and is a consequence of the fact that the orbifold twist in this case belongs to the center of the duality group.

We can understand the existence and the location of these minima more succinctly following the discussion section 2.3 in a way that will be generalized to other twist groups in section 4.4. Every monodromy $\mathcal{M}_n$ of finite order n has $|Tr(\mathcal{M}_n)| < 2$ and is in the elliptic $\mathrm{SL}(2,\mathbb{R})$ conjugacy class $\mathcal{M}_n$, so that it is conjugate to the rotation matrix $\mathcal{M}_e$ given in (3.1), for some value of the angle of rotation m. Moreover, since $(\mathcal{M}_n)^n = \mathbb{1}$, the angle must be $m = \frac{2\pi N}{n}$ for some integer N. The monodromies in (3.5) are in fact conjugate to the rotation matrix R_n, where R_n is the $\mathrm{SO}(2)$ rotation through $\frac{2\pi}{n}$, i.e. there exists a (constant) $\mathrm{SL}(2,\mathbb{R})$ matrix S_n such that

$$S_n \mathcal{M}_n S_n^{-1} = R_n \tag{4.5}$$

Note that given an S_n that solves this equation, left-multiplication by an arbitrary $\mathrm{SO}(2)$ matrix k gives another matrix $S'_n \equiv kS_n$ that also solves this equation. We can use this gauge freedom to bring all matrices S_n to an upper triangular form. For the cases $n = 2, 3, 4, 6$, the matrices S_n are given by

$$S_2 = V\,, \qquad S_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad S_3 = S_6 = \sqrt{\frac{2}{\sqrt{3}}}\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} \end{pmatrix}. \tag{4.6}$$

Note that S_2 is an arbitrary $\mathrm{SL}(2,\mathbb{R})$ upper triangular matrix V because M_2 depends on an arbitrary $\mathrm{SL}(2,\mathbb{R})$ matrix A which can written as a product $A = kV$ where k is an $\mathrm{SO}(2)$ matrix.

For these monodromies, the mass matrix M_n can be chosen (using the ambiguity discussed in section 2.2) so that after this conjugation it becomes the rotation generator

$$S_n M_n S_n^{-1} = \frac{2\pi}{n}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{4.7}$$

We have seen in section 2.3 that for such a mass matrix, the Scherk-Schwarz potential has a global minimum at $\mathcal{V} = S_n$ at which the potential vanishes, and that this is a fixed point under the action of the twist group generated by $\mathcal{M}_n$.

We thus conclude that for an elliptic duality twist $\mathcal{M}_n \in \mathrm{SL}(2,\mathbb{Z})_\tau$, the critical points of the Scherk-Schwarz potential are precisely at the fixed points of the twist. The potential vanishes at the minimum and the theory at the minimum is a symmetric orbifold of the type discussed above using the twist group generated by $\mathcal{M}_n$ accompanied by a shift.

Orbifolds with twisted boundary conditions around toroidal directions have been considered before, for example, in [41, 30, 42, 43], usually with boundary conditions that break supersymmetry. Our analysis illuminates the place of such orbifold conformal field theories in the string configuration space. If we climb up the Scherk-Schwarz potential from the minimum, the string equations of motion will no longer be satisfied and there would be no CFT description of the theory because we have perturbed the CFT by an irrelevant

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perturbation. Nevertheless, from the spacetime point of view, it is a mild way of going off-shell with operators that correspond to massive fields in spacetime with masses of order of the inverse radius of the circle and our analysis gives the off-shell potential.

Duality twists that belong to $\mathrm{SL}(2,\mathbb{Z})_\sigma$ are related to the one above by a T-duality along one of the legs of the torus. The most general case when we twist by an arbitrary element of $O(2,2;\mathbb{Z})$ would therefore twist the coordinate and the T-dual coordinate independently of each other. The minima of the potential in this case would be described by the most general order n asymmetric orbifold with an asymmetric rotation of the torus accompanied by a shift along the circle.

The possible asymmetric rotations can be easily classified [33] and are given by the automorphisms of the lorentzian lattice $\Gamma^{2,2}$ for special values of the moduli that are left fixed by the twists. There are fixed planes for the cases that we have already discussed when the T-duality twist acts only on τ or only on σ. There are also fixed points in the general case that have more symmetry. For example, the point $\sigma = i, \tau = i$ has an enhanced $(\mathbb{Z}_4 \times \mathbb{Z}_4) \rtimes \mathbb{Z}_2$ symmetry, the point $\sigma = \tau = \rho$ with $\rho = e^{\pi i/3}$ has an enhanced $\mathbb{Z}_9$ symmetry and the point $\sigma = i$, $\tau = \rho$ (or vice versa) has a $\mathbb{Z}_{12}$ symmetry which acts quasicrystallographically [31] on the lattice. At any of these points in the moduli space, a $\mathbb{Z}_n$ subgroup of the symmetry can be combined with an order n shift to obtain an asymmetric orbifold. This orbifold would describe the theory at the minimum of the potential in the corresponding Scherk-Schwarz reduction, with mass matrix of the form (3.11).

4.2 Superstrings

In the case of superstrings, the action of the orbifold rotation must be lifted to spacetime fermions. Consider, for example, a $\mathbf{T}^2$ reduction along the X^8 and X^9 directions. The torus coordinate z can be written as $X^8 + iX^9$, and the $\mathbb{Z}_n$ rotations discussed in the previous section are generated by elements $\exp(2\pi i J_{89}/n)$ where J_{89} is the generator of rotations in the 89 plane. When spacetime fermions are present, the eigenvalues of J_{89} are half-integral and $\exp(2\pi i J_{89}) = (-1)^F$ where F is the fermion number; as a result these rotations now generate $\mathbb{Z}_{2n}$ groups of order $2n$. For odd n, an order n symmetry generated by $\exp(2\pi i J_{89}/n)(-1)^F$ is also possible. We suppose there is a further circular direction X^7 say, and orbifold by these transformations combined with the appropriate shifts in the X^7 coordinate.

These orbifolds break supersymmetry completely because in the light cone Green-Schwarz formalism (with X^8, X^9 both transverse coordinates), no components of Spin(8) spinors are left invariant by the rotation in the 89 plane. When the radius of the X^7 circle is of string scale, all these models contain tachyons in the twisted sector and are unstable. However, for a large enough circle there will be no tachyons and the twisted states will be very massive. This is the regime in which one can compare the orbifolds with the supergravity analysis of compactifications with duality twists given in the previous sections.

The above applies to orbifolds based on subgroups of $\mathrm{SL}(2,\mathbb{Z})_\tau$. For the heterotic string, the ones based on $\mathrm{SL}(2,\mathbb{Z})_\sigma$ are related by T-duality and are very similar. For the type-IIA string, orbifolds by subgroups of $\mathrm{SL}(2,\mathbb{Z})_\sigma$ are distinct from orbifolds by subgroups of $\mathrm{SL}(2,\mathbb{Z})_\tau$ and are T-dual to orbifolds of type-IIB by subgroups of $\mathrm{SL}(2,\mathbb{Z})_\tau$.

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When the duality twist does not belong to the T-duality group then the theory at the minimum of the Scherk-Schwarz potential cannot be described by a perturbative orbifold, but the supergravity analysis of section 2 and section 3 is still applicable. For example, in the supergravity analysis the twists that correspond to turning on Ramond-Ramond fluxes are on the same footing as those that correspond to turning on NS-NS fluxes (see below for a discussion of fluxes in this context). The group theoretic considerations of this and the previous sections can be equally well applied to such nonperturbative twists, in particular for finding the minima of the Scherk-Schwarz potential.

For the standard reduction on $\mathbf{T}^2$ followed by a twisted reduction on $\mathbf{S}^1$ of type-IIB, all nonperturbative twists belong to SL(3). If we restrict attention to the nonperturbative $\mathrm{SL}(2,\mathbb{Z})_\lambda$, then the considerations are similar to those for $\mathrm{SL}(2,\mathbb{Z})_\sigma$. The monodromy $\mathcal{M}_2$ actually corresponds to a perturbative symmetry $\Omega(-1)^{F_L}$ where Ω is orientation reversal and F_L is the left-moving fermion number [44]. Therefore, modding out by this symmetry gives rise to a perturbative orientifold. The orientifold has no orientifold planes or D-branes because of the shift along the circle. The $\mathcal{M}_3, \mathcal{M}_4, \mathcal{M}_6$ twists are nonperturbative and the Scherk-Schwarz potential will fix the dilaton-axion field λ to either i or $e^{\pi i/3}$ where the string would be strongly coupled. The classical analysis given here can still be reliable in such situations in the spirit of F-theory [45], especially if the theory at the minimum preserves enough supersymmetry. Since this $\mathrm{SL}(2,\mathbb{Z})_\lambda$ is conjugate to $\mathrm{SL}(2,\mathbb{Z})_\sigma$ by an element of SL(3) we expect that the theories at the minima with nonperturbative twists will be dual to the perturbative orbifolds discussed above by using the adiabatic argument [46].

It is easy to construct models with unbroken supersymmetries by compactifying on higher tori of dimensions $2N$ and choosing a duality twist that is a subgroup of $\mathrm{SU}(N)$. The resulting orbifold theory at the minimum then has $\mathrm{SU}(N)$ holonomy and preserves some number of supersymmetries. As a simple example that illustrates this point, consider type-IIB on a $\mathbf{T}^4 \times \mathbf{S}^1$. We take the twists to be in $\mathrm{SL}(4,\mathbb{Z})$ which is the group of large diffeomorphisms of $\mathbf{T}^4$. The simplest nontrivial conjugacy class is the element $-\mathbb{1}$ that generates a twist group of order two. Because it is a twist of finite order, the Scherk-Schwarz potential will have a stable minimum and the $\mathbb{Z}_2$ symmetry of the orbifold theory at the minimum is generated by the reflection of all coordinates of $\mathbf{T}^4$ accompanied by a half-shift along the circle. Note that without the half-shift, the $\mathbf{T}^4/\mathbb{Z}_2$ orbifold would have given us a $\mathbf{K}_3$ and we would have obtained a standard type-IIB compactification on $\mathbf{K}_3 \times \mathbf{S}^1$ to five dimensions with sixteen unbroken supersymmetries. When the orbifolding action includes the half-shift, one would still obtain a theory in five dimensions with sixteen supersymmetries, but all twisted states will now be massive. In particular, the vector multiplets that come from the sixteen fixed points of the reflection on $\mathbf{T}^4$ will now be massive thereby stabilizing all moduli that belong to these multiplets as well as the moduli in the untwisted sector that are projected out by the orbifolding.

4.3 Relation to turning on fluxes

In this subsection we explain the relation between the twisted reductions and compactifications with internal fluxes.

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The toroidal compactification on $\mathbf{T}^2$ followed by this twisted reduction on an $\mathbf{S}^1$ is equivalent to reducing on a three-manifold B which is the total space of the torus bundle over a circle with metric

$$ds_B^2 = (2\pi R)^2 dy^2 + \frac{\mathcal{A}}{\tau_2}|dx_1 + \tau(y)dx_2|^2 \tag{4.8}$$

where the fiber is a $\mathbf{T}^2$ with real periodic coordinates x_1, x_2, $x_i \sim x_i + 1$, constant area modulus $\mathcal{A}$ and complex structure $\tau(y)$, which depends on the coordinate y. The twisted reduction on the circle with the ansatz $\tau(y) = \tau_{g(y)}$ associated with a particular torus bundle B is precisely the compactification on the three dimensional total space B [8]. For the parabolic conjugacy class, $\tau(y) = \tau_1 + i\tau_2 + ny$ where m is the integral mass parameter in (3.6), and τ_1, τ_2 are independent of y, x_i. Then the metric is

$$ds_B^2 = (2\pi R)^2 dy^2 + \frac{\mathcal{A}}{\tau_2}(dx_1 + A)^2 + \mathcal{A}\tau_2 dx_2^2 \tag{4.9}$$

where $A = (\tau_1 + ny)dx_2$. The total space can also be regarded as a circle bundle over a 2-torus [8], with fiber coordinate x_1, base space coordinates y, x_2 and connection 1-form A and first Chern number n. We thus see that the parabolic conjugacy class $\mathcal{M}_{T_n}$ corresponds to turning on n units of magnetic flux of the Kaluza-Klein gauge field. T-dualizing in the x_1 fiber direction untwists the bundle to give a torus metric on $\mathbf{T}^3$

$$ds_B^2 = (2\pi R)^2 dy^2 + \frac{\tau_2}{\mathcal{A}} dx_1^2 + \mathcal{A}\tau_2 dx_2^2 \tag{4.10}$$

but turns on a B-field with field strength $H = n dx_1 \wedge dx_2 \wedge dy$ corresponding to a constant H-flux over $\mathbf{T}^3$.

For the elliptic conjugacy classes, the orbifold at the minimum of the potential can be viewed as turning on magnetic flux tubes similar to the non-compact Melvin solutions [47, 48, 49]. In the non-compact Melvin solution, the orbifolding action is a rotation in a plane accompanied by a shift along a circle and this orbifold can be interpreted as a Melvin background with magnetic flux of the Kaluza-Klein vector potential. The total flux in the plane is a function of the angle of rotation in the plane and since the angle is continuous, the flux can be changed continuously. By contrast, in the situation that we discuss in this paper, the rotation angle is quantized because we are rotating the coordinate of a torus and not of a plane. As we have seen, the only allowed rotation angles for $\mathbf{T}^2$ are $\pi/3$, $\pi/2$, π, and $2\pi/3$ and consequently only a finite number of discrete values of the flux are allowed.

For the hyperbolic cases, the situation is more complicated and it is unclear whether there is a relation of the reduction to a toroidal reduction with flux.

4.4 Generalizations

Generalizations to higher duality groups are very interesting and can be used to fix moduli in a more realistic context preserving some supersymmetry. We will not analyze explicit models here but instead present a number of general results that are useful for the analysis of the Scherk-Schwarz potential in these cases.

We consider a theory with a moduli space $G(\mathbb{Z})\backslash G/K$ with G non-compact semi-simple and K the maximal compact subgroup.[1] Our prime example will be $G = \mathrm{SL}(N,\mathbb{R})$ and $K = \mathrm{SO}(N)$.

For $G(\mathbb{Z})$ (e.g. $\mathrm{SL}(N,\mathbb{Z})$), many more conjugacy classes are possible and we will not discuss them explicitly here. One general question of interest for a given conjugacy class is whether the Scherk-Schwarz potential has a minimum, and if so, where in the moduli space it lies. The following theorem is useful for addressing this question. See, for example, [50] for a proof.

Theorem. *Every finite order subgroup $H \subset G(\mathbb{Z}) \subset G$ with G non-compact semi-simple is conjugate to a subgroup of the maximal compact subgroup K. Thus, there exists a matrix $S \in G$ such that $SHS^{-1} = K_1 \subset K$.*

The space G/K is defined as a coset with the equivalence relation $g \sim kg$ for every $g \in G$ and $k \in K$. If we denote the equivalence class of g by $[g]$ then the coset is the set $\{[g]\}$ of all equivalence classes. The equivalence class of the identity $[\mathbb{1}]$ corresponds to the entire group K. An element h of G acts on the coset by right multiplication $[g] \to [gh]$. It is clear from the equivalence $\mathbb{1}K = K\mathbb{1} \sim \mathbb{1}$ that the point $[\mathbb{1}]$ in G/K is a fixed point under the action of K by right multiplication. Therefore, by the theorem above, every finite order subgroup H also has a fixed point on G/K. This property is closely related to the fact that the spaces G/K have negative curvature. Indeed, the equivalence class $[S]$ is the desired fixed point under right-multiplication by H since $SH = K_1S \sim S$. It is also clear that since $k^tk = \mathbb{1}$, the metric $\mathcal{H}_0 = S^tS$ is invariant under H-transformations: $h^tS^tSh = S^tS$ for all $h \in H$. Because H leaves the metric invariant, it defines a symmetry of the corresponding integer lattice in $\mathbb{R}^N$ and can be used for orbifolding.

These results imply that any twist $\mathcal{M}$ that generates a finite order subgroup H is conjugate by an $\mathrm{SL}(N,\mathbb{R})$ matrix S to an $\mathrm{SO}(N)$ matrix. By (2.3), it will result in a mass matrix that is conjugate by S to a rotation generator. We have seen in section 2.3 that in this case when mass matrix is conjugate to a rotation generator, $\mathcal{V}_0 = S$ or $\mathcal{H}_0 = S^tS$ is a stable minimum of the Scherk-Schwarz potential. Using this physics input we conclude that for the finite order twists $H \subset \mathrm{SL}(N,\mathbb{Z})$ the matrix S defines a minimum on the coset of the Scherk-Schwarz potential at which $V = 0$.

5. Conclusions

Even though we have focused here on duality twists in $\mathbf{T}^2 \times \mathbf{S}^1$ compactifications, these methods can be applied equally well to more general compactifications on higher tori and other manifolds such as **K3** and Calabi-Yau threefolds that have interesting duality symmetries. We have seen that there is a close relation between compactifications with perturbative duality twists and orbifolds. Our considerations here are useful even for nonperturbative duality twists and for duality twists that correspond to turning on internal RR-fluxes.

[1] Because K acts on the left and $G(\mathbb{Z})$ on the right in our conventions in this paper, the coset should be denoted by $K\backslash G/G(\mathbb{Z})$; however, with a slight abuse of notation, we adhere to the common usage, denoting the moduli space by $G(\mathbb{Z})\backslash G/K$.

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The structure of duality twists for higher groups is expected to be much richer because many more conjugacy classes are possible. For general twists, the Scherk-Schwarz potential can be quite complicated and explicit extremization is not easy. However, the group theoretic considerations discussed here provide an efficient way for finding the minima and the properties of the theory at the minima. It would be interesting to elucidate further the relation of duality twisting with compactifications with internal fluxes and to see if some of the recent models that fix moduli with fluxes can be analyzed in this framework.

We have seen that in the type-II circle compactifications considered here with SL(2) twists, only the elliptic conjugacy classes lead to stable minima. However, in more general toroidal compactifications with higher groups, it is likely that other conjugacy classes also lead to stable minima. For example, the parabolic conjugacy classes correspond to turning on H-flux. It is known that in orientifolds of type-I on $\mathbf{T}^6$, if additional orientifold charges are present, the inclusion of 3-form fluxes can lead to gauged supergravities [51, 52] that have stable minima [23, 26]. It would also be interesting to see in the more general cases which twists lead to stable minima. In such more general situations, the twist groups may have fixed sub-manifolds instead of fixed points in the moduli space where the potential has a minimum. In such cases, only some of the moduli will be stabilized.

By considering a U-duality twist that has a unique fixed point on the moduli space, one can construct models with or without supersymmetry in any dimension that stabilize all moduli except the radius of the circle used for twisting. In the framework described here we require an $\mathbf{S}^1$ factor for twists but in more general situations where the manifold of compactification has circle fibration, it might be possible to twist along this fiber in a way analogous to F-theory [45, 53, 54]. If supersymmetry is broken, the classical analysis would be quantum corrected but we expect that the existence and the location of the minima which depend on considerations of symmetry should still be valid. It would be interesting to explore further if these different techniques can be combined to construct realistic models with few or no moduli.

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